On approximation measures of certain q-continued fractions

Ville Merilä

University of Oulu, Finland

Abstract

Let $\mathbb{K}$ be an algebraic number field of degree κ over $\mathbb{Q}$. Let

$$||*||_v = |*|_v^{\kappa_v/\kappa}, \quad \kappa_v = [\mathbb{K}_v : \mathbb{Q}_v],$$

be the normalized valuation of $\mathbb{K}$ and denote $\lambda = \lambda_q = \log H(q)/\log ||q||_v$, where $H(q)$ is the height of $q \in \mathbb{K}^*$ satisfying $|q|_v < 1$. Then the (proper) q-continued fraction

$$G(q) = \mathbf{K}_{n=1}^{\infty} \frac{q^{s(n-1)}(S_0 + S_1 q^{n-1} + \ldots + S_h q^{h(n-1)})}{T_0 + T_1 q^n + \ldots + T_l q^{ln}}, \; S_i, T_i \in \mathbb{K}, \; S_0 T_0 \neq 0,$$

where $s \geq 1$ and $s + \lambda A > 0$, has an approximation measure (exponent) $\mu = s\kappa/\kappa_\nu(s + A\lambda)$ where $A = \max\{l, (s+h)/2\}$.

The results imply, for example, irrationality measures $\mu = 3/(3 + 2\lambda)$ for the famous Ramanujan-Selberg continued fractions

$$S_1(q) = \frac{(-q^2;q^2)_\infty}{(-q;q^2)_\infty} = \frac{1}{1+}\frac{q}{1+}\frac{q^2+q}{1}\,\frac{}{+}\frac{q^3}{1+}\frac{q^4+q^2}{1}\,\frac{}{+\ldots},$$

$$S_2(q) = \frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty} = \frac{1}{1+}\frac{q+q^2}{1}\,\frac{}{+}\frac{q^4}{1+}\frac{q^3+q^6}{1}\,\frac{}{+}\frac{q^8}{1+\ldots}$$

and for Eisenstein's continued fraction

$$E_1(q) = \sum_{n=0}^{\infty} q^{n^2} = \frac{1}{1-}\frac{q}{1-}\frac{q^3-q}{1}\,\frac{}{-}\frac{q^5}{1-}\frac{q^7-q^3}{1}\,\frac{}{-\ldots},$$

related to the Jacobi Theta functions.